Some quick notes about a more expressive quantification logic

What follows are just some quick thoughts ive had about the issue since writing my brief comments in the review of SEP's principle of compositionality article.

In that article i stumbled upon some sentences that cud seemingly not be adequately formalized by first-order predicate logic, or many-order predicate logic. Not even with identity and modal logical extensions. At least, so it seemed to me. I dont know how to prove such things.

So, it immediately seemed to me that i shud invent some way to formalize it. Surely, it cud not be so difficult. Im quite used to making up logic systems by now, having experimented with erotetic logic (of questions+answers), and imperative logic (have been working on an essay about that, but not much work seems to be necessary), and some other modal logic systems (f.i. the one able to handle multiple different kinds of possibilities at once).

Anyway, the sentences were (renamed):

> (1) Most students will succeed if they work hard.
> (2) Most students who work hard will succeed.[1]

It seems to me that predicate logic cannot adequately formalize notions of "most". Predicate logic only works with 0, >0 and all. Two of these are also indefinite quantifiers, but not as tricky as "most".

Well, in a trivial sense, one can sort of formalize such sentences, but it will not be pretty. One can start by listing an infinite list of conditions about how many students there are. If the number is 3, then one can formalize (1) as something like:

> (1')* (∃!x)(∃!y)(∃!z)((WorkHard(x)∧WorkHard(y)∧WorkHard(z))→(Succeed(x) ∧Succeed(y)).

In logic english: There are three unique students, if they all work hard, then two of them will succeed.

This is equivalent with the above when the number of students is exactly 3. However, for those that know how i think about equivalence and identify, it will not surprise u that i dont consider (1')* to be identical in meaning with (1), even when the number is 3. They are merely equivalent in my view. They have a different propositional structure, cf. Swartz and Bradley 1979.

So, how does one otherwise formalize it? There shud be some general way so as to avoid the nasty feature of having to have an infinite series of conditions in front which chooses the right equivalent formalization.

I thought of using sets from math, and it seems to work elegantly. First, we extend predicate logic to allow quantifiers to range over set variables, represented by greek capital letters (those that dont look like latin letters, to avoid confusion).
Then extend with notions and symbols for set membership (∈), and subset (⊆), and cardinality (#), and division (/). Also extend to the natural numbers (at least). One might need identify between sets as well, but not for these two particular formalizations.

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[1] For those who have trouble grasping the difference, and it is somewhat elusive, see the picture representations in the appendix.

Here’s what i came up with.

(1)* $(\exists\Gamma)(\exists\Delta)(\forall x)(\forall y)((\mathrm{Student}(x)\leftrightarrow x\in\Gamma)\wedge\Delta\subseteq\Gamma\wedge\#\Delta>(\#\Gamma/2)\wedge((y\in\Delta)\rightarrow(\mathrm{WorkHard}(y)\rightarrow\mathrm{Succeed}(y))))$

In logic english: There are two sets (gamma and delta), and for any two things (x and y), x is a student iff x belongs to gamma, and delta is a subset of gamma, and the cardinality of delta is more than half the cardinality of gamma, and if y belongs to delta, then if y works hard, then y will suceed.

Quite a mouth full. But it is not so complex. The reason for using the two sets is that we need to capture the meaning of “most”. This is done by having the one set be a subset of the other, and having it be at least half as big. The inclusion criteria for belonging to the superset is just being a student. So, all the members of the subset are also students. Then, for most of the students, that is, all members of the subset, if they work hard, then they will all succeed. And we’re done.

The other formalization is handed in a similar way, just changed appropriately to reflect the change in the wording.

(2)* $(\exists\Gamma)(\exists\Delta)(\forall x)(\forall y)(((\mathrm{Student}(x)\wedge\mathrm{WorkHard}(x))\leftrightarrow x\in\Gamma)\wedge\Delta\subseteq\Gamma\wedge\#\Delta>(\#\Gamma/2)\wedge(y\in\Delta\rightarrow\mathrm{Succeed}(y)))$

In logic english: There are two sets (gamma, delta), and for any two things (x,y), if x is a student and x works hard, then x belongs to gamma, and delta is a subset of gamma, and the cardinality of delta is more than half of the cardinality of gamma, and if y belongs to delta, then y will suceed.

So, the same cardinality trick is used to capture the notion of “most”.

Simplification of formalization?
Right now the formalization is annoyingly long, even if it is accurate. One can perhaps simplify the formalization with some shorthands.
To begin with, one can define the range for x and y to be students, and thus get rid of the student predicate. This helps only a little.

The annoying part to get rid of is “$\Delta\subseteq\Gamma\wedge\#\Delta>(\#\Gamma/2)$”, and perhaps also “$x\in\Gamma$” and “$y\in\Delta$”. The first can be simplfied as something like most(Δ,Γ) which is just a shorthand for the above. Perhaps like this:

(1)** $(\exists\Gamma)(\exists\Delta)\mathrm{most}(\Delta,\Gamma)(\forall x)(x\in\Gamma)(\forall y\in\Delta)(\mathrm{WorkHard}(y)\rightarrow\mathrm{Succeed}(y)$

Logic english: there are two sets (gamma and delta), delta is a subset of gamma, and most members of gamma are members of delta, and for any student, x, that student belongs to gamma, and for any student that belongs to delta, if that student works hard, then he will succeed.

Somewhat better. But how about:

(1)*** $(\exists\Gamma)(\exists\Delta)\mathrm{most}(\Delta y,\Gamma x)(\forall x)(\forall y)(\mathrm{WorkHard}(y)\rightarrow\mathrm{Succeed}(y)$

Logic english: there are two sets, gamma and delta, the most relation holds between delta,y and gamma,x, and for any x and y, if y works has, then y will succeed.

The idea is that the belonging of the members is now included in the most relation.
It now holds all the machinery that make formalization of “most” quantifiers possible.
Unpacking “most(Δy,Γx)” yields: $(\Delta\subseteq\Gamma\wedge\#\Delta>(\#\Gamma/2))\wedge(\forall x)(x\in\Gamma)\wedge(\forall y)(y\in\Delta)$. The universal quantifiers signaling that all x’s belong to gamma, and all y’s belong to delta is now completely implicit.

The last step is making the existence of the sets implicit as well:

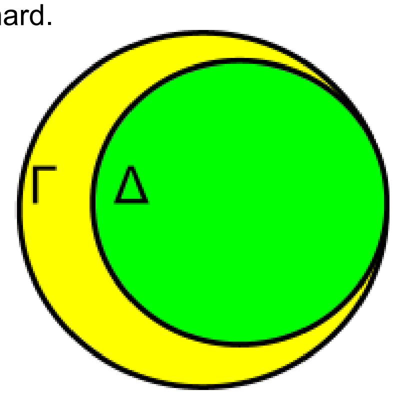
(1)**** most(Δy,Γx)(∀x)(∀y)(WorkHard(y)→Succeed(y)

However, does this simplification still allow us to formalize (2) properly?

(2)**** most(Δy,Γx)(∀x)(∀y)(WorkHard(x)→Succeed(y)

Seems so.

Appendix
(1) Most students will succeed if they work hard.
If those students in the green circle work hard, then they will all succeed. Nothing is stated about the ones in the yellow cirle who are not in the green. Perhaps they are not even working hard.



(2) Most students who work hard will succeed.
If everybody works hard, then those in the green circle will succeed. No information is given about what happens to those in the outer category. Perhaps they will succeed as well, perhaps they won’t, perhaps results will be mixed. Everybody works hard.